Optimum Compression of a Gas by a Cylindrical Shock Wave in Presence of a Magnetic Field

a- Awaneesh Jee Srivastava**,** **b**- S.N.Ojha

a-   Department of Mathematics

SRMGPC, Lucknow

awaneeshcancerian@gmail.com

b-Department of Mathematics, S.C.P.G.College, Ballia

sheonarainojha@gmail.com

1. ABSTRACT

Simple analytical solutions have been obtained for cylindrical converging shock waves in an ideal perfectly conducting gas in presence of a magnetic field. These solutions are derived with the assumption that a cylindrical piston drives a shock front to the axis of the cylinder at a critical moment of cumulating. Simple particular solutions have been also derived.

Key Words   Shock Waves, High Temperature, Thermonuclear, Magnetic field, Cylindrical Piston.

2. INTRODUCTION

The concentration or compression of force, energy or any other physical quantity in a small volume is an important phenomenon occurring in nature. A considerable role is played by concentric waves of cylindrical or spherical symmetry which generate high parameter physical fields about the axis or centre of cumulation. Among the other applications, it is viewed as necessary step in realizing controlled thermonuclear fusion. The converging shock waves are also employed in laboratories to maintain high temperature to study various processes which take place in gases.

The problem of compression shocks was investigated first by Guderley [1]. Guderley considered a spherical or cylindrical converging shock wave, which was created formally at time $t = -\infty$ by a spherical or cylindrical piston at a distance far from the centre or axis of cumulation, that reaches to the centre or axis at a moment $t = 0$. The self-similar solution of the problem was solved numerically for the situation with the initial

push (piston) which does not influence on the final stages of cumulation. Further, this problem of cumulation

was solved by Stanyukovich [2], Zeldovich and Raizer [3], Goldman [4] and many others by using only

numerical computations. Chisnell [5], unlike previous investigations was able to get accurate analysis and

numerical results and to yield a simple analytic description of flow variables at all points behind the converging

shocks.

The optimum compression of an ideal gas by a cylindrical piston was considered by Zhdanov and

Tribnikov [6] without forming any shock waves. In that case the sound wave fronts move with a sound velocity

in front of the piston and this non-shock wave problem also needs numerical computations. Similar problem has

been considered and simple analytical solutions have been derived by Farshi and Trubnikov [7] with the

assumption that the piston moves with a supersonic velocity and therefore, a shock wave front exists in front of

the piston.

Here, we have assumed that the piston drives a shock wave front to the axis of cylinder at the critical

moment of cumulating of an ideal gas in presence of a magnetic field. Simple analytical solutions are derived

and particular cases have been discussed.

3. BASIC EQUATIONS AND SHOCK CONDITIONS

Consider a cylindrical piston which instantaneously begins to converge with supersonic speed at time $t =$

0 and drives a shock front so that it reaches to the axis at time $t = t_*$. In the region between the front and the

piston one may consider the equations of motion in cylindrical symmetry as (Vishwakarma and Yadav [8])

$$(3.1) \qquad \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \left(\frac{\partial v}{\partial r} + \frac{v}{r} \right) = 0,$$

$$(3.2) \qquad \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{h}{\rho} \frac{\partial h}{\partial r} = 0,$$

$$(3.3) \qquad \frac{\partial (p \rho^{-\gamma})}{\partial t} + v \frac{\partial (p \rho^{-\gamma})}{\partial r} = 0,$$

(3.4)
$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial r} + h\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) = 0.$$

These equations govern the mass density ρ, the velocity v, the pressure p, the axial magnetic field h, all of which depend only on the time t and on the distance r from the axis of the cylinder, γ is the ratio of specific heats of the gas.

From (3.1) and (3.4), we may obtain

(3.5)
$$\frac{h}{\rho} = \text{constant}.$$

Equation (3.5) holds only when the gas is perfectly conducting and magnetic field is axial in the case of cylindrically symmetric flow.

Taking pressure in the form (Golitsyn [9], Ojha and Singh [10], Vishwakarma and Yadav [8]),

(3.6)
$$p = \beta\, p^*, \qquad p^* = p + \frac{h^2}{2} = \text{total pressure}, 0 < \beta < 1,$$

and using (2.5), the effective speed of sound c is defined by

(3.7)
$$c^2 = \frac{dp^*}{d\rho} = a^2 + b^2 = \frac{\gamma^* p^*}{\rho},$$

(3.8)
$$a^2 = \frac{\gamma p}{\rho} \text{ is square of speed of sound,}$$

(3.9)
$$b^2 = \frac{h^2}{\rho} \text{ is the square of the Alfven speed}$$

and

(3.10)
$$\gamma^* = \gamma\,\beta + 2(1 - \beta).$$

Assuming β to be a constant and flow to be isentropic, we may write

(3.11)
$$\frac{\partial(p^*\rho^{-\gamma^*})}{\partial t} + v\frac{\partial(p^*\rho^{-\gamma^*})}{\partial r} = 0.$$

The equation (3.11) is true only when (3.5) holds and β has a constant value. The assumption that β is constant, physically means that the gas pressure and the magnetic pressure are in constant ratio.

From (3.11) we may write,

(3.12) $p^* = A \rho^{\gamma^*}, \quad A = \text{constant}$

and

(3.13) $c^2 = A \rho^{(\gamma^*-1)}.$

Equations (3.1) and (3.2) may be written as

(3.14) $$\frac{\partial c^2}{\partial t} + v\frac{\partial c^2}{\partial r} + (\gamma^* - 1)\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) = 0,$$

and

(3.15) $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{(\gamma^* - 1)}\frac{\partial c^2}{\partial r} = 0.$$

We assume that at time t, in between t = 0 and t = t$_*$, the cylindrical piston with radius $R_p(t)$ has a supersonic velocity $V_p(t)$ and therefore, a shock wave front moves with radius $R_f(t)$ and velocity $V_f(t)$ in the region ahead of the piston. Again, let just ahead of the shock front ($r < R_f$) the mass density, pressure, temperature, magnetic field strength and velocity of the gas be ρ_0, p_0, T_0, h_0 and $v_0 = 0$ respectively, whereas just behind of the shock front ($R_f < r < R_p$) these quantities be ρ_1, p_1, T_1, h_1 and v_1 correspondingly. The well known Hugoniot conditions on the shock wave front may be written as

(3.16) $\rho_1(V_f - v_1) = \rho_0 V_f,$

(3.17) $p_1^* + \rho_1(V_f - v_1)^2 = p_0^* + \rho_0 V_f^2,$

(3.18) $$\frac{1}{2}(V_f - v_1)^2 + \frac{\gamma p_1}{\rho_1(\gamma - 1)} + \frac{h_1^2}{\rho_1} = \frac{1}{2}V_f^2 + \frac{\gamma p_0}{\rho_0(\gamma - 1)} + \frac{h_0^2}{\rho_0}.$$

Taking $\xi = 1 - (\rho_0/\rho_1)$, we may write these conditions as

(3.19) $v_1 = \xi V_f,$

(3.20) $p_1^* - p_0^* = \rho_0 V_f^2 \xi = \rho_0 V_f v_1,$

$$(3.21) \qquad p_1{}^* = p_0{}^* \left\{ \frac{2\beta + \xi(\gamma - 1)}{2\beta - \xi(2\beta + \gamma - 1)} \right\}.$$

The velocity of the shock wave front V_f may be written as

$$(3.22) \qquad v_1 \, V_f = \left\{ \frac{2\xi(\beta + \gamma - 1)}{2\beta - \xi(2\beta + \gamma - 1)} \right\} \frac{p_0^*}{\rho_0},$$

or

$$(3.23) \qquad V_f{}^2 = \left\{ \frac{2(\beta + \gamma - 1)}{2\beta - \xi(2\beta + \gamma - 1)} \right\} \frac{p_0^*}{\rho_0}.$$

Putting $\quad c_0{}^2 = \dfrac{\gamma * p_0^*}{\rho_0} = \dfrac{[\gamma\beta + 2(1 - \beta)]p_0^*}{\rho_0} \quad$ in (3.23), where c_0 is effective speed of sound in front of

the shock, we have

$$(3.24) \qquad V_f{}^2 = \frac{2(\beta + \gamma - 1)c_0{}^2}{\{\gamma\beta + 2(1 - \beta)\}\{2\beta - \xi(2\beta + \gamma - 1)\}}$$

or

$$(3.25) \qquad V_f{}^2 - \frac{(2\beta + \gamma - 1)}{2\beta} \xi V_f{}^2 = \frac{(\beta + \gamma - 1)c_0{}^2}{[\gamma\beta + 2(1 - \beta)]\beta},$$

which gives

$$(3.26) \qquad V_f = G + \surd(C_0{}^2 + G^2),$$

where

$$(3.27) \qquad G = \left(\frac{2\beta + \gamma - 1}{4\beta} \right) v_1$$

and

$$(3.28) \qquad C_0{}^2 = \frac{(\beta + \gamma - 1)c_0{}^2}{\beta(\gamma\beta + 2(1 - \beta))}.$$

4. SOLUTION BEHIND THE SHOCK FRONT

Substituting the relations

$$(4.1) \qquad v = v_r \ = -\alpha \left(\frac{r}{t_* - t} \right)$$

and

$$(4.2) \qquad c = \delta \left(\frac{r}{t_* - t} \right)$$

into the equations (3.14) and (3.15), we observe that when

$$(4.3) \qquad \alpha = \frac{1}{2 + \beta(\gamma - 2)}$$

and

$$(4.4) \qquad \delta = \frac{\beta(\gamma - 2) + 1}{\sqrt{2}\,(2 + \beta(\gamma - 2))}$$

equations (3.14) and (3.15) are satisfied. Thus, we can say that (3.1) and (3.2) are simplest solutions of the flow

equations with constants α and δ given by equations (3.3) and (3.4).

For such solutions the motion of the piston is described by the equations,

$$(4.5) \qquad \frac{dR_p}{dt} = v_{(r=R_p)} = -\frac{\alpha R_p}{t_* - t}, \qquad R_p = R_0 \left(1 - \frac{t}{t_*} \right)^{\alpha}$$

and

$$(4.6) \qquad V_p = -V_0 \left(1 - \frac{t}{t_*} \right)^{\alpha - 1}$$

where $R_0 = R_p$ at $t = 0$, $V_0 = \alpha R_0 / t_*$. After time $t = 0$, the shock wave front moves towards the axis with a

constant velocity, it means $V_f = - R_0 / t_*$ which is derived from

$R_f = R_0 \left(1 - \dfrac{t}{t_*}\right)$. The velocity of the gas just behind the front will be given by

(4.7)$\qquad v_1 = v_{(r=R_f)} = -\dfrac{\alpha R_f}{t_* - t} = -\alpha R_0/t_* = -V_0 = \text{constant}.$

By substituting these values into the relations (3.26) we get

(4.8)$\qquad \dfrac{C_0^{\ 2}}{V_f^{\ 2}} = \left(1 - \dfrac{\alpha(2\beta + \gamma - 1)}{2\beta}\right)$

or

(4.9)$\qquad \dfrac{c_0^{\ 2}}{V_f^{\ 2}} = \dfrac{\beta\{\gamma\beta + 2(1-\beta)\}}{(\beta + \gamma - 1)}\left(1 - \dfrac{\alpha(2\beta + \gamma - 1)}{2\beta}\right)$

and

(4.10)$\qquad \dfrac{C_0^{\ 2}}{v_1^{\ 2}} = \dfrac{1}{\alpha}\left(1 - \dfrac{\alpha(2\beta + \gamma - 1)}{2\beta}\right)$

or

(4.11)$\qquad \dfrac{c_0^{\ 2}}{v_1^{\ 2}} = \dfrac{\beta\{\gamma\beta + 2(1-\beta)\}}{\alpha(\beta + \gamma - 1)}\left(1 - \dfrac{\alpha(2\beta + \gamma - 1)}{2\beta}\right)$

5. PARTICULAR CASES

Taking $\gamma = 5/3$, we have

(5.1)$\qquad \alpha = \dfrac{3}{6 - \beta}$

and

(5.2)$\qquad \delta = \dfrac{3 - \beta}{(6 - \beta)\sqrt{2}}$

For a particular value of $\beta = 0.5$, we obtain $\alpha = 6/11$ and $\delta = 5/11\sqrt{2}$. In this case the law of the piston motion

has the following form,

$$(5.3) \qquad R_p = R_0 \left(1 - \frac{t}{t_*} \right)^{6/11}$$

If the radius R_0 and the sound velocity c_0 are given, then the critical moment t_* will be given by

$$(5.4) \qquad t_* = \frac{R_0}{c_0 \sqrt{14}}$$

The jump in density and pressure before and behind the front are given by

$$(5.5) \qquad \frac{\rho_1}{\rho_0} = \frac{1}{1 - \alpha} = \frac{11}{5}$$

and

$$(5.6) \qquad \frac{p_1^*}{p_0^*} = 15$$

Similarly, for other constant values of β ($0 < \beta < 1$), one may derive other particular solutions. These particular solutions are suitable only for the initial stages of compression. Therefore, for the final stages of compression it is necessary to use the Guderley's numerical solutions. These solutions have only the analytical values as the simplest solutions.

REFERENCES

[1] Guderley, G.   :   Luftarhtforschung, 19, 302, (1942).

[2] Stanyukovich, K. P.   :   Unsteady motion of continuous media,

                                       Pergamon (1960).

[3] Goldman, E. B.   :   Plasma Phys. 15, 289, (1973).

[4] Zeldovich, Y. B.   :   Physics of Shock Waves and High

    and Raizer, Yu. P.   :   Temperature Hydrodynamic Phenomena.

                                       Vol. II, Academic Press (1967).

[5] Chisnell, R. F.   :   Fluid Mech. 354, 357, (1998).

[6] Zhdanov, S. K. and   :   JETP Lett. 21, 371, (1975).

    Trubnikov, B. A.

[7] Farshi, E. and   :   Fusion Engg. Design, 60, 99-103, (2002).

    Trubnikov, B. A.

[8] Vishwakarma, J. P.   :   Eur. Phy. J. B, 34, 247, (2003).

    and Yadav, A. K.

[9] Golitsyn, G. S.   :   Soviet Phys., JETP, 35, 538, (1959).

[10] Ojha, S. N. and   :   IL Nuovo Cimento, 70B, 2, 212, (1982).

    Singh, S. N.